A Generalized Pythagoras Theorem Using Natural Induction Over the Dimensions.

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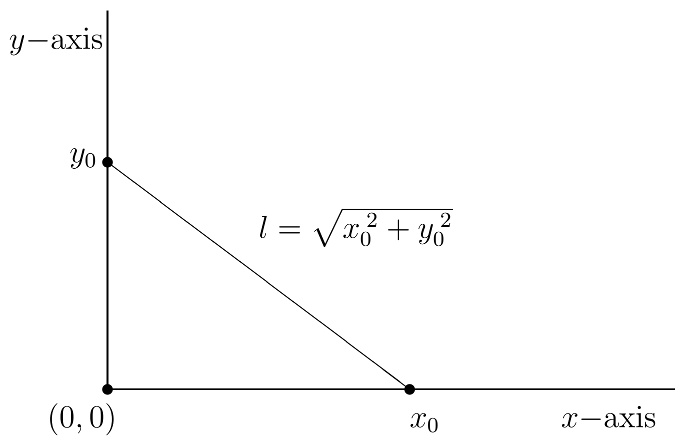
The 3D theorem of Pythagoras using the 2D theorem of Pythagoras.

We start with the 2D theorem of Pythagoras, I think if you are reading on this website it is rather likely that you have seen enough proofs for this:

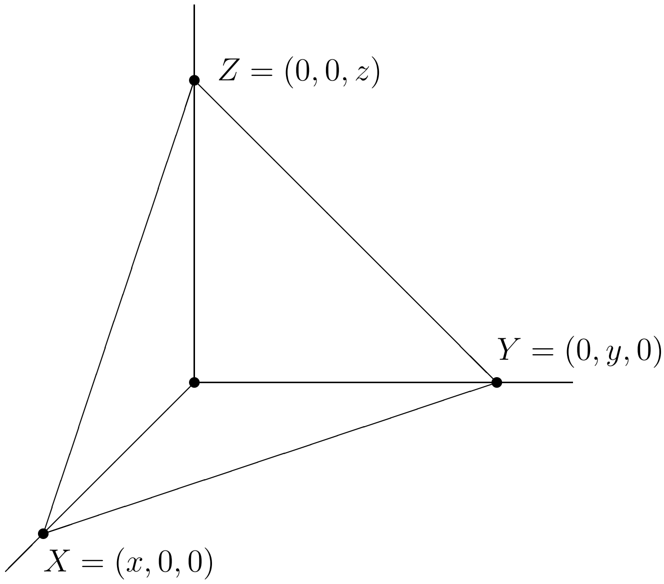


With 3 of such rectangles we are going to make a pyramid as next:
For positive x, y and $z \in \mathbb{R}$
we define 3 points as next:
$X = (x, 0, 0)$,
$Y = (0, y, 0)$ and
$Z = (0, 0, z)$.
Together with the origin O this forms a rectangular pyramid $OXYZ$ with four faces.
The four triangular faces are subject to the 3D version of the Pythagorean theorem:
If a, b and c denote the area of the

three triangles with a right angle in it and
d denotes the area of $\triangle XYZ$, in
that case we have:

$$a^2 + b^2 + c^2 = d^2.$$

And it looks more or less like this in a 3D
natural basis setting:



If we project the pyramid $OXYZ$ along the $z-$axis on
the $xy-$plane we denote that as $\hat{Z}$. So $\hat{Z} = \triangle OXY$.
In a similar fashion:
$\hat{X} = \triangle OYZ, \hat{Y} = \triangle OXZ$ and $\hat{O} = \triangle XYZ$.

Of course we also want the projection $\hat{O}$ to be
perpendicular onto $\triangle XYZ$, so we have to project
the stuff along the normal vector of $\triangle XYZ$. Because
we need normal vectors like this in the general
setting in $\mathbb{R}^n$ we give the $\mathbb{R}^3$ easy to
understand normal vector to $\triangle XYZ$:

$$\vec{N} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

Normalization of $\vec{N}$ gives us the normal vector of unit length:

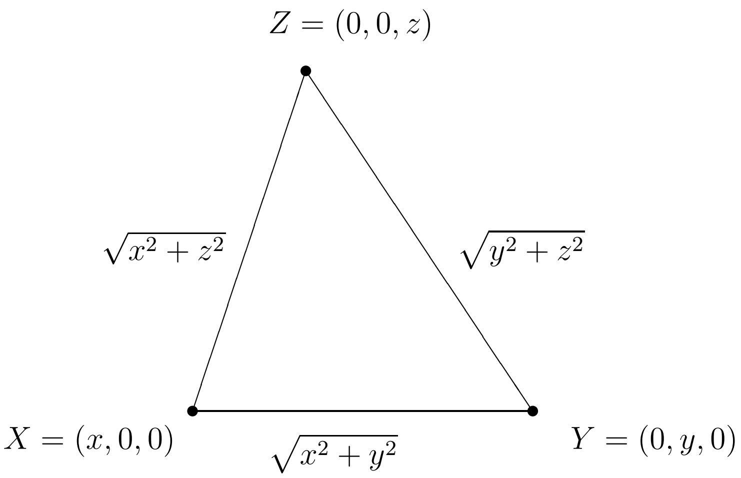
$$\vec{n} = \frac{1}{\sqrt{(yz)^2 + (xz)^2 + (yz)^2}} \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

It is **very easy** to prove that $\vec{n}$ is actually a normal vector: the three points X, Y and Z are also vectors and the inner product with $\vec{n}$ yields always the same value:

$$< \vec{n}, X >=< \vec{n}, Y >=< \vec{n}, Z >= \frac{xyz}{\sqrt{(yz)^2 + (xz)^2 + (xy)^2}}$$

Of course elementary linear algebra dictates that this value is the distance to the origin O abundantly demonstrating that $\vec{n}$ is indeed a normal vector for the plane going though $\triangle XYZ$.

Beside ultra cute, the next picture is very worthy of taking a second thought about it:
It portraits a triangle who's sides are clearly ruled by the 2D theorem of Pythagoras:



The reader is invited to check that the area of the above triangle $\triangle XYZ$ is given by

$$a = \frac{1}{2}\sqrt{(yz)^2 + (xz)^2 + (xy)^2}\ .$$

Now we are going to calculate this area a for ourselves and please look again at the first picture of this post:
The area of any triangle is given by the half of the bottom line and the height. If we use the XY line as the 'bottom line' it is obvious this has a length of $\sqrt{x^2+y^2}$.

Furthermore if d denotes the distance from the line segment XY to the origin, it is obvious the height of the triangle $\triangle XYZ$ is given by

$$h = \sqrt{d^2+z^2}.$$

And finding d is also very easy, a normal vector to the line segment XY is given by:

$$\vec{N} = \begin{pmatrix} y \\ x \end{pmatrix}, \quad \text{normalizing this gives } \vec{n} = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} y \\ x \end{pmatrix}.$$

We observe

$$< X, \vec{n} > = < Y, \vec{n} > = \frac{xy}{\sqrt{x^2+y^2}}$$

Hence the height h of $\triangle XYZ$ is given by:

$$h = \sqrt{\frac{(xy)^2}{x^2+y^2} + z^2}.$$

Now we have all the ingredients in place to calculate the area a of the triangle $\triangle XYZ$ via calculating $\|XY\| \cdot h/2$. Here we go using the fact that $\|XY\| = \sqrt{x^2+y^2}$:

$$a = \frac{1}{2}\sqrt{x^2+y^2}\sqrt{\frac{(xy)^2}{x^2+y^2} + z^2} = \frac{1}{2}\sqrt{(xy)^2+(xz)^2+(yz)^2}.$$

A few definitions and the natural induction step.

We now turn to the general case on $\mathbb{R}^n$ and we will observe the so called induction step where we need to show that if the Pythagoras theorem holds for some value n, it will also hold for $n+1$.

In $\mathbb{R}^n$ we can use the same notation as we did in $\mathbb{R}^3$ but instead of a triangle stuff like that is known as a simplex. A simplex is just a convex combination of $n+1$ points in $\mathbb{R}^n$.

For our goal we need a lot of perpendicular stuff therefore in $\mathbb{R}^n$ we take the origin O as the top of our simplex.
Instead of the general word simplex it is better to view them as generalized rectangular triangles (or rectangular pyramids from $\mathbb{R}^3$).

We can write such a convex combination as follows: $[O, X_1, X_2, \cdots, X_n]$.
This is a set of points given by the convex combinations like:

$$[O, X_1, X_2, \cdots, X_n] =$$

$$\{X \in \mathbb{R}^n | X = t_0 O + t_1 X_1 + \cdots + t_n X_n, \sum_{k=0}^{n} t_k = 1 \text{ and } \forall k : t_k \geq 0\}.$$

Of course the term $t_0 O$ does not contribute very much but I leave it in because the above is more or less the general definition of a simplex.

We also have all kinds of perpendicular projections onto the sides of such a convex combination. For example $\hat{O}$ denotes the projection of $[O, X_1, X_2, \cdots, X_n]$ on $[X_1, X_2, \cdots, X_n]$. So we can write or define $\hat{O}$ to be:

$$\hat{O} := [X_1, X_2, \cdots, X_n] =$$

$$\{X \in \mathbb{R}^n | X = t_1 X_1 + \cdots + t_n X_n, \sum_{k=1}^{n} t_k = 1 \text{ and } \forall k : t_k \geq 0\}.$$

The other n projections along the n coordinate axis are simply denoted as $\hat{X}_i$ and we simply leave out X_i from making any contribution to the convex sum or convex combination:

$$[O, X_1, \cdots, X_{i-1} X_{i+1}, \cdots, X_n] =$$

$$\{X \in \mathbb{R}^n | X = t_0 O + t_1 X_1 + \cdots + t_n X_n, \sum_{k=0,\ k \neq i}^{n} t_k = 1 \text{ and } \forall k : t_k \geq 0\}.$$

Now inside $\mathbb{R}^n$ is a generalized pyramid $[O, X_1, \cdots, X_n]$ also has a n

dimensional volume. We use the standard Lebesgue volume measurement and an n dimensional measurement will be denoted as λ_n. Of course for a rectangular generalized pyramid (or a simplex where at one point all stuff meets perpendicular) is given by:

$$\lambda_n([O, X_1, \cdots, X_n]) = \frac{1}{n!}\, x_1 x_2 \cdots x_n.$$

But the general theorem of Pythagoras is a statement about the $n+1$ faces (projections) like
$\hat{X}_i = [O, X_1, \cdots, X_{i-1} X_{i+1}, \cdots, X_n]$
and these volumes need to be measured with λ_{n-1} and cannot be measured with λ_n.
That is logical since, for example in $\mathbb{R}^3$ a plane or a segment of a 2D plane never has 3D volume.

The $n-1$ dimensional volume $\lambda_{n-1}(\hat{X}_i)$ is given by:

$$\lambda_{n-1}(\hat{X}_i) = \lambda_{n-1}([O, X_1, \cdots, X_{i-1}, X_{i+1}, \cdots X_n]) = \frac{1}{(n-1)!}\, x_1 \cdots x_{i-1} x_{i+1} \cdots x_n.$$

So the thing we **desire to prove** is:

$$\lambda_{n-1}(\hat{O})^2 = \sum_{k=1}^{n} \lambda_{n-1}(\hat{X}_k)^2$$

The 'hat notation' like in $\hat{O}$ is of course orthogonal or perpendicular projection and as such the left hand side $\lambda_{n-1}(\hat{O})$ is the hard nut to crack.
But on the right hand side, all projections $\hat{X}_k$ still have the origin in them and there all things important meet perpendicular.

In order to keep stuff readable it is very handy to give a separate name to the $n-1$ dimensional volumes of those projected faces. Let it be p_k for projection number k: So the $n-1$ dimensional volume is a cakewalk (gluten free of course):

$$p_k := \lambda_{n-1}(\hat{X}_k) = \frac{1}{(n-1)!}\, x_1 x_2 \cdots x_{k-1} x_{k+1} \cdots x_n.$$

So the factor x_k is projected out...
Using the projection volume numbers p_i the general theorem of Pythagoras would look like the next:

$$\lambda_{n-1}(\hat{O})^2 = \sum_{k=1}^{n} p_k^2.$$

After so much blah blah blah I hope you did not fall asleep because we are now going to make a normal vector for the hyperplane through $\hat{O}$.
Recall that the projection $\hat{O}$ is the convex span of n points X_k where each X_k is lying somewhere on the positive part in the direction of the e_k basis vector of the natural basis.
So what is a good basis vector?
Utterly simple, the next one will do:

$$\vec{N} = \begin{pmatrix} x_2 x_3 \cdots x_n \\ x_1 x_3 \cdots x_n \\ \vdots \\ \vdots \\ x_1 \cdots x_{n-2} x_n \\ x_1 \cdots x_{n-2} x_{n-1} \end{pmatrix} = (n-1)! \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix}$$

Why is this a valid normal vector?
Once more it is utterly simple, taking the inner product of any of the points X_k with this normal vector $\vec{N}$ always returns the same value namely:

$$< X_1, \vec{N} >=< X_2, \vec{N} >= \cdots =< X_n, \vec{N} >= x_1 x_2 \cdots x_n.$$

From elementary linear algebra we know that it is a pretty standard feature that inside a (hyper) plane all points give the same value when the inner product with a normal vector is taken...
Hence $\vec{N}$ is perpendicular to $\hat{O}$.

A normal vector to $\hat{O}$ of unit length is easy to construct, with this unit vector we can

find an expression for the distance of $\hat{O}$
to the origin O:

$$\vec{n} = \frac{1}{\sqrt{p_1^2 + \cdots + p_n^2}} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix}.$$

If we denote this distance by d we get, for example by taking the inner product with X_1

$$d := d(\hat{O}, O) =< X_1, \vec{n} >= \frac{x_1 x_2 \cdots x_n}{\sqrt{p_1^2 + \cdots + p_n^2}}$$

After all this work we can finally **take the induction step!** And all this work was only needed because we were in a general setting on $\mathbb{R}^n$...

We suppose the theorem of Pythagoras has been proven for some fixed n. As a matter of fact above we did prove it for $n = 3$ using the fact that for thousands of years it was already proved for $n = 2$.
So now we take the $n - 1$ dimensional volume of $\hat{O}$, we expand $\mathbb{R}^n$ with one extra dimension to $\mathbb{R}^{n+1}$. So we add one more perpendicular coordinate axis and a point X_{n+1} with coordinate value x_{n+1}.
And after that all we do is multiply $\lambda_{n_1}(\hat{O})$ by the distance between $\hat{O}$ and X_{n+1}.

This all **perfectly the same as the 3D case**:
With h we denote the distance between the new added point X_{n+1} and $\hat{O}$, we get a new $\hat{O}$ that I will denote as $\hat{O}^+$ and it's value n dimensional value equals:

$$\lambda_n(\hat{O}^+) = \frac{1}{n+1} \sqrt{d^2 + x_{n+1}^2} \cdot \lambda_{n-1}(\hat{O})$$

Formulating d^2 in terms of the p_k^2 and
the same for $\hat{O}$ gives:

$$\lambda_n(\hat{O}^+) = \frac{1}{n} \cdot \frac{1}{(n-1)!} \cdot \sqrt{\frac{(x_1 \cdots x_{n+1})^2}{p_1^2 + \cdots + p_n^2} + (n-1)!^2 x_{n+1}^2} \cdot \sqrt{p_1^2 + \cdots + p_n^2}.$$

Once more we are now calculating inside $\mathbb{R}^{n+1}$ and the projection $\hat{O}^+$ is a n dimensional simplex, therefore we must multiply by the leading factor of $1/n$.

I have to admit that the $(n-1)!^2$ can make the reader a bit confused but this 'coupling constant' $(n-1)!^2$ is needed because I made the choice that the p_k numbers represent the volume of the projected faces.

If you use a number scheme with the p_k being simply $x_1 x_2 \cdots x_n / x_k$, in that case you would not have this strange looking coupling constant but I can assure you strange looking stuff will pop up in other places.

If we work out the two square root signs, what we get under the sqrt sign looks like this:

$$(x_1 \cdots x_{n+1})^2 + (n-1)!^2 x_{n+1}^2 \sum_{k=1}^{n} p_k^2 =$$

$$\sum_{k=1}^{n+1} (x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n+1})^2.$$

Basically here you have your generalized theorem of Pythagoras, if you multiply by $(1/n!)$ you get:

$$\lambda_n(\hat{O}^+)^2 = \frac{1}{n!} \sum_{k=1}^{n+1} (x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n+1})^2.$$

Inside $\mathbb{R}^n$ we denoted the projected volumes as p_k where it was supposed that the projection was done via the e_k coordinate axis. We expanded $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ and instead of naming it p_k let's name it q_k where now k runs from 1 to $n+1$.

The projected volume q_k values are given by:

$$q_k = \frac{1}{n!} x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n+1}.$$

And the induction step now says:

$$\lambda_n(\hat{O}^+)^2 = q_1^2 + q_2^2 + \cdots + q_{n+1}^2.$$

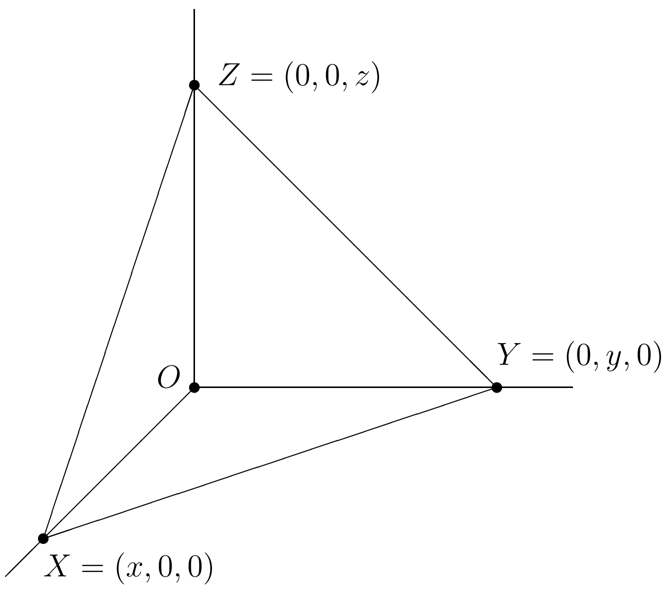
A very strange sphere based on the 3D theorem of Pythagoras.

I know I know in the last post I wrote that it would be the final post on the general theorem of Pythagoras...

But in that last post I could not use the polygraph applet from the WIMS collection of applets so I was forced to leave that out.
Yet I found another website where the WIMS stuff is not only up and running; miracle miracle it even plots a graph when I fill in a cute formula!

So you must view this post more as an addendum that was not posted in the previous update.

Let us look once more at that triangle from two posts back that served as an introduction to the three dimensional version of the theorem of Pythagoras:



Now we wonder what the squares of the areas of the three triangles with a right angle in it at O will add up to if we keep it fixed.
In previous posts we denoted the triangle $\triangle OXY$ as $\hat{Z}$. The hat notation like in $\hat{Z}$ only means that

this convex set $[OXYZ]$ is projected into the $z-$axis direction.
If we denote the area of $\triangle OXY$ by the two dimensional volume measurement $\lambda_2(\hat{Z})$, in that case the 3D version of the Pythagorean theorem says that:

$$\lambda_2(\hat{O})^2 = \lambda_2(\hat{X})^2 + \lambda_2(\hat{Y})^2 + \lambda_2(\hat{Z})^2.$$

Since the area's of the three rectangular triangles $\hat{X}, \hat{Y}$ and $\hat{Z}$ are just basic high school stuff:

$$\begin{cases} \lambda_2(\hat{X}) = yz/2 \\ \lambda_2(\hat{Y}) = xz/2 \\ \lambda_2(\hat{Z}) = xy/2 \end{cases}$$

I hope you are not confused by all this notation with hats and lambda's but how does the graph of

$$\lambda_2(\hat{O})^2 = 1 \text{ look like?}$$

Here is how it looks when I cap the absolute value of the coordinate values at 2:

And for capping it off at a distance of 6 it looks like this:

Why does this graph avoid the coordinate axes?
Very simple, for example on the $x-$axis both the y and z coordinates are zero and as such an expression like $(yz)^2 + (xz)^2 + (xy)^2$ will always add up to zero.

I sincerely hope it is not that hard to understand that

$$\lambda_2(\hat{X})^2 + \lambda_2(\hat{Y})^2 + \lambda_2(\hat{Z})^2 = \left(\frac{yz}{2}\right)^2 + \left(\frac{xz}{2}\right)^2 + \left(\frac{xy}{2}\right)^2.$$

And that for triangles $\triangle XYZ$ to have an area of 1, it is a cakewalk (gluten free!) to state that this means:

$$\left(\frac{yz}{2}\right)^2 + \left(\frac{xz}{2}\right)^2 + \left(\frac{xy}{2}\right)^2 = 1 \text{ or } (yz)^2 + (xz)^2 + (xy)^2 = 4.$$

Is stuff like this a 'sphere'?
Is stuff like this useful into crafting a new kind of norm where the length of a triple (x, y, z) is defined by:

$$\|(x, y, z)\| = \sqrt{\left(\frac{yz}{2}\right)^2 + \left(\frac{xz}{2}\right)^2 + \left(\frac{xy}{2}\right)^2}.$$

It does not satisfy the triangle equation for normed spaces that says for a norm to be useful it should satisfy the fact that in a triangle the length of any side is shorter than the sum of the other two sides:

$$\|X + Y\| \leq \|X\| + \|Y\|.$$

As a matter of fact, all points that are on one of the three coordinate axis have a length of zero under this strange norm and usually we want just one vector to be the zero thing.

So I hope you take this title of 'strange sphere' with a grain of salt; it might look as some kind of sphere equation but in the practice of day to day math it just does not work properly...
For every positive value r in

$$\left(\frac{yz}{2}\right)^2 + \left(\frac{xz}{2}\right)^2 + \left(\frac{xy}{2}\right)^2 = r,$$

the thing is unbounded while if we take $r = 0$ we observe that the solution is made up of the three coordinate axis. With that cute detail we are at the end of this post, till updates.

An example of an idea that did not work.
A long long time ago when our solar system and the planet earth were far far away from where we are now, I tried to find the five dimensional exponential curve.

So I tried to do the same that brought success to finding a parametrization of the exponential circle in $\mathbb{R}^3$: use the cosine and a few time lags so that the exponential circle goes through the three basis vectors.

In the five dimensional space $\mathbb{R}^5$ the 5D numbers can be written as:

$$X = x_0 + x_1 l + x_2 l^2 + x_3 l^3 + x_4 l^4 \text{ with } l^5 = \pm 1.$$

The basic idea was to start the first coordinate function as $f_0(t) = \cos(t)$, the second

coordinate function has a time lag of $2\pi/5$
and is $f_1(t) = \cos(t - 2\pi/5$. Etc etc
until the last coordinate function $f_4(t) = \cos(t - 8 * \pi/5)$.
That was just a try out for crafting the next
'exponential curve' (again; the idea did not
float):

$$f(t) = f_0(t) + f_1(t)l + f_2(t)l^2 + f_(t)3l^3 + f_4(t)l^4 \text{ with } l^5 = \pm 1.$$

I was on the right track because for some
strange reason the sum of squares adds
up to 5/2 or 2.5.
It we denote this sum
of squares as $k(t)$, so

$$k(t) = \cos^2(t) + \cdots + \cos^2(t - 8\pi/5).$$

Just some experimental evidence
in the picture below:

The main reason this does not work lies, for
example, in the fact that for $t = 2\pi/5$ the
second coordinate function equals 1 while
the others are non-zero.
That means it does not go through the second
basis vector because after all in the second
basis vector all other coordinates are zero:
$(0, 1, 0, 0, 0)$....

It was later in that year that I found the
modified Dirichlet kernels, they do the
trick and satisfy all equations possible
that define the exponential curve.
Here is a picture of the first time delay
so the second coordinate function:

For $\mathbb{R}^5$ the first coordinate function
is given as the modified kernel:

$$f_0(t) = \frac{\sin 5t}{5 \sin t}$$

Remark the period in time is π so the
first time lag is given by

$$f_1(t) = f_0(t - \pi/5) = \frac{\sin 5(t - \pi/5}{5 \sin(t - \pi/5)}$$

Teaser for the $\tau = \log l$ post.

In the next post I will show you a general method to calculate the logarithm of the first imaginary unit. This will be done in $\mathbb{R}^7$ and we will be using the circular multiplication. Seven dimensional circular numbers are always of the form

$$X = x_0 + x_1 l + x_2 l^2 + \cdots + x_6 l^6 \quad \text{with } l^7 = 1.$$

And we are going to calculate what $\log l$ is, it is hard work and in these pictures I only show you some results from $\mathbb{R}^3, \mathbb{R}^5$ and of course $\mathbb{R}^7$.

Here are the numerical values for $\mathbb{R}^3$, the circular multiplication:

$$\log \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1.2092 & 1.2092 \\ 1.2092 & 0 & -1.2092 \\ -1.2092 & 1.2092 & 0 \end{pmatrix}.$$

How to find expressions for that numerical value 1.2092? Of course if you are visiting this website longer you might already know

$$\log j = \tau = \frac{2\pi}{3\sqrt{3}}(j - j^2).$$

Remark that $2\pi/3\sqrt{3} \approx 1.2092$.
I sincerely hope that by now you know that $f(t) = e^{\tau t}$ is the 3D version of the exponential circle in the complex plane $\mathbb{C}$ that is given by $f(t) = e^{it}$.

In January 2015 I was able to find a good representation in $\mathbb{R}^5$, here are the results from an internet applet. Can you find expressions for the observed numerical values?

$$\log \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -1.068959 & 0.660653 & -0.660653 & 1.068959 \\ 1.068959 & 0 & -1.068959 & 0.660653 & -0.660653 \\ -0.660653 & 1.068959 & 0 & -1.068959 & 0.660653 \\ 0.660653 & -0.660653 & 1.068959 & 0 & -1.068959 \\ -1.068959 & 0.660653 & -0.660653 & 1.068959 & 0 \end{pmatrix}.$$

And now it is just much harder to find the desired stuff. But I tried the ratio between the first and second imaginary component, that is:

$$\frac{1.068959}{-0.660653} \approx -1.6180 \ \text{ holy cow! } = -\frac{1+\sqrt{5}}{2}.$$

And once I had this understandable ratio between the two pairs of imaginary numbers I was able to crack the τ for $\mathbb{R}^5$.

But the method did not work for higher dimensions like $\mathbb{R}^7$, all I could do was look at the numbers while not able to find them myself...

Yet later that year while I was riding my bicycle through the Harener swamp I thought of a completely different method for the situation on $\mathbb{R}^3$.

It worked and at the time I understood this different approach that leans heavily on diagonal matrices could be a method for any dimension.

So here are the values for the log of the first imaginary component in $\mathbb{R}^7$, circular multiplication (I only give you the first column because that contains all the information we need):

$$\log l = \log \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1.034376 \\ -0.574035 \\ 0.460341 \\ -0.460341 \\ 0.574035 \\ -1.034376 \end{pmatrix}.$$

Now the result of the next post will be that you can indeed use matrix diagonalization for finding these numbers.

$$\begin{cases} \Im_1\, \tau = 2\pi/7^2 \cdot (2\sin 2\pi/7 + 4\sin 4\pi/7 + 6\sin 6\pi/7) & \approx 1.034376055 \\ \Im_2\, \tau = 2\pi/7^2 \cdot (2\sin 4\pi/7 - 4\sin 6\pi/7 - 6\sin 2\pi/7) & \approx -0.574035403 \\ \Im_3\, \tau = 2\pi/7^2 \cdot (2\sin 6\pi/7 - 4\sin 2\pi/7 + 6\sin 4\pi/7) & \approx 0.460340651 \end{cases}$$

Electron acceleration by one Volt of potential difference.

The most common energy unit these days is the Joule but for small particles like the electron they invented the electron Volt or eV.

If an electron with initial zero kinetic energy moves through an electric potential of 1 Volt it has gained one eV in kinetic energy.

One eV = 1.6×10^{-19} Joules, the kinetic energy is given as usual as

$$E_k = \frac{1}{2}\, mv^2.$$

If we plug in the electron mass of about 9.1094×10^{-31} we get the next:

$$1.6 \times 10^{-19}\ \text{J/kg}\ = \frac{1}{2}\, 9.1094 \cdot 10^{-31} v^2,$$

for the velocity v this gives:

$$v = \sqrt{\frac{1.6 \times 10^{-19}}{0.5 \times 9.1094 \times 10^{-31}}}\ \text{m/s}\ \approx 600\ \text{km/s}.$$

So a speed of 600 kilometer per second for just going through one Volt in potential difference... With that in the back of your mind it suddenly is understandable that electron acceleration by magnetic fields has been missed for a full century...

The general inverse Pythagoras theorem.

I can't remember that I ever observed the so called inverse Pythagoras theorem. The 2D version is given by the relation

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{d^2}$$

Modify the next pic...