Pythagoras' Theorem for Areas—Revisited

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1 Introduction

In [4] an n-dimensional analog of the Pythagorean Theorem is formulated and proved—involving $n-1$ dimensional areas, and not lengths. The authors came across the three-dimensional version "incidentally," and only subsequently learned of its history. Indeed, they found the n-dimensional version predates their paper, originating in [3]. I also happened on the three-dimensional version of the theorem "incidentally," in [2], where the following remarks appear:

> During the first quarter of the seventeenth century both René Descartes (1596–1650) and his somewhat older contemporary, John Faulhaber (1580–1635), came across the trirectangular tetrahedron, that is, the tetrahedron $OABC$ such that the three face angles of one of its trihedral angles, say O, are all right angles. Both of them knew the property of such a tetrahedron which is the analog of the Pythagorean theorem, namely, that the square of the area of the face opposite the vertex O of the "right angle" is equal to the sum of the squares of the areas of the other three faces.

The proof for the n-dimensional case in [4] is direct and straightforward. The authors note that in older books of geometry the three-dimensional version was sometimes proved as an application of vector products. In 1964 I too formulated and proved an n-dimensional analog—my proof, in fact, begins by generalizing the notion of vector product to n dimensions. Since this alternative approach may be of some independent interest, I present it here.

2 Terminology and Background

In $\mathbb{R}^n$, $m(\leq n)$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ determine the analog of a parallelogram. It consists of all vectors of the form $t_1\mathbf{v}_1 + \cdots + t_m\mathbf{v}_m$ with $0 \leq t_i \leq 1$. We refer to it as the m-dimensional *parallelepiped* determined by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. According to the classic [1], the square of the (m-dimensional) volume of this parallelepiped is the determinant $|AA^T|$, where A is the matrix with the coordinates of $\mathbf{v}_i$ in row i.

We can also think of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ as determining the analog of a triangle. This consists of all vectors of the form $t_1\mathbf{v}_1 + \cdots + t_m\mathbf{v}_m$ where $t_i \geq 0$ and $\sum t_i = 1$. We refer to this as the m-dimensional *tetrahedron* determined by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. The ($m$-dimensional) volume of the tetrahedron determined by $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is $1/m$ times the volume of the parallelepiped determined by them.

3 Cross Products, Generalized

The notion of cross product (or vector product) in $\mathbb{R}^3$ is a standard topic in calculus books. For vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ the cross-product $\mathbf{a}\times\mathbf{b}$ is sometimes defined to be the vector whose magnitude is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$, and with direction orthogonal to the plane containing $\mathbf{a}$ and $\mathbf{b}$ (given by what is sometimes called the 'right-hand rule'). There is a second characterization: the vector $\mathbf{a}\times\mathbf{b}$ is given by the following determinant.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

In evaluating this determinant the usual rules are followed formally—real numbers are multiplied; numbers times unit vectors are treated as scalar multiplication.

All this is familiar stuff in three dimensions. As it happens, there is a natural analog for higher dimensions. In $\mathbb{R}^{n+1}$ think of a cross product as a combination of n vectors.

Definition 3.1 For vectors $\mathbf{v}_1 = \langle v_{1,1}, \ldots, v_{1,n}, v_{1,n+1}\rangle$, …, $\mathbf{v}_n = \langle v_{n,1}, \ldots, v_{n,n}, v_{n,n+1}\rangle$ in $\mathbb{R}^{n+1}$, the cross product is

$$\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle\!\rangle = \begin{vmatrix} \mathbf{e}_1 & \ldots & \mathbf{e}_{n+1} \\ v_{1,1} & \ldots & v_{1,n+1} \\ \vdots & & \vdots \\ v_{n,1} & \ldots & v_{n,n+1} \end{vmatrix}$$

where $\mathbf{e}_1$, …, $\mathbf{e}_{n+1}$ are the unit vectors $\langle 1, 0, \ldots, 0\rangle$, …, $\langle 0, 0, \ldots, 1\rangle$ respectively.

Proposition 3.2 *Let $\boldsymbol{v}_1$, …, $\boldsymbol{v}_n$ be as above.*

1. *For a vector $\boldsymbol{w} = \langle w_1, \ldots, w_{n+1}\rangle$*

$$\boldsymbol{w}\cdot\langle\!\langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle\!\rangle = \begin{vmatrix} w_1 & \ldots & w_{n+1} \\ v_{1,1} & \ldots & v_{1,n+1} \\ \vdots & & \vdots \\ v_{n,1} & \ldots & v_{n,n+1} \end{vmatrix}.$$

2. *For an orthogonal matrix (transformation) T,*

$$\langle\!\langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle\!\rangle T = \langle\!\langle \boldsymbol{v}_1 T, \ldots, \boldsymbol{v}_n T \rangle\!\rangle.$$

Proof Item 1 is immediate by the definitions of cross and inner products. For item 2 it is enough to show that $\mathbf{w}\cdot[\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle\!\rangle T] = \mathbf{w}\cdot\langle\!\langle \mathbf{v}_1 T, \ldots, \mathbf{v}_n T\rangle\!\rangle$ for any vector $\mathbf{w}$, since then projections on elements of a basis will be the same. Without loss of generality we can take $\mathbf{w}$ to be $\mathbf{u}T$. Now by part 1, and the properties of orthogonal matrices that $|T| = 1$ and T preserves inner products, we have the following.

$$\begin{aligned}
\mathbf{u}T \cdot \langle\!\langle \mathbf{v}_1 T, \ldots, \mathbf{v}_n T \rangle\!\rangle &= \left| \begin{bmatrix} \mathbf{u}T \\ \mathbf{v}_1 T \\ \vdots \\ \mathbf{v}_n T \end{bmatrix} \right| = \left| \begin{bmatrix} \mathbf{u} \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} T \right| \\
&= \left| \begin{bmatrix} \mathbf{u} \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} \right| |T| = \left| \begin{bmatrix} \mathbf{u} \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} \right| \\
&= \mathbf{u} \cdot \langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle\!\rangle = \mathbf{u}T \cdot [\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle\!\rangle T]
\end{aligned}$$

■

The following says the cross product generalization also generalizes the three dimensional definition based on area.

Proposition 3.3 *In $\mathbb{R}^{n+1}$, $\langle\!\langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle\!\rangle$ is orthogonal to each $\boldsymbol{v}_i$, and the magnitude of $\langle\!\langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle\!\rangle$ is equal to the n-dimensional volume of the parallelepiped determined by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.*

Proof The orthogonality of $\mathbf{v}_i$ and $\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle\!\rangle$ is immediate from part 1 of Proposition 3.2.

Part 2 of Proposition 3.2, and the fact that orthogonal transformations preserve lengths, combine to say that the length of a generalized cross product is preserved under orthogonal transformations. Consequently in showing the result connecting magnitudes and volumes we can assume that unit vector $\mathbf{e}_{n+1}$ is orthogonal to each of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, since we can always rotate about the origin to effect this state of affairs. Then each $\mathbf{v}_i$ has an $n+1$st component of 0, and consequently

$$\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle\!\rangle = \begin{vmatrix} \mathbf{e}_1 & \ldots & \mathbf{e}_n & \mathbf{e}_{n+1} \\ v_{1,1} & \ldots & v_{1,n} & 0 \\ \vdots & & \vdots & \vdots \\ v_{n,1} & \ldots & v_{n,n} & 0 \end{vmatrix} = (-1)^n \begin{vmatrix} v_{1,1} & \ldots & v_{1,n} \\ \vdots & & \vdots \\ v_{n,1} & \ldots & v_{n,n} \end{vmatrix} \mathbf{e}_{n+1}.$$

The conclusion now follows using the result mentioned in Section 2, from [1], concerning volumes of parallelepipeds. ■

4 Generalized Pythagorean Theorem

The space is $\mathbb{R}^{n+1}$. O is the origin. Pick $n+1$ points $A_1, \ldots, A_{n+1}$, one on each axis, so that $\vec{OA_i} = a_i \mathbf{e}_i$ where $a_i > 0$. These $n+1$ vectors determine an $n+1$-dimensional tetrahedron T. The vertex of T at O is the analog of a right angle. T has $n+2$ faces, which are n dimensional—each face is determined by n vectors of the form $\vec{OA_i}$. (Picturing this with $n+1 = 3$ may be of use.) Call the face that does not contain the origin the *hypotenuse face*. The following is from [4].

Theorem 4.1 *The square of the n dimensional volume of the hypotenuse face of T is equal to the sum of the squares of the n dimensional volumes of the other $n+1$ faces.*

Proof The n dimensional parallelepiped determined by $\vec{OA_1}, \ldots, \vec{OA_{i-1}}, \vec{OA_{i+1}}, \ldots, \vec{OA_{n+1}}$ has an n-dimensional analog of a right angle at the origin, and so its n-dimensional volume is $a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n+1}$. Then the n-dimensional volume of the face of T determined by $\vec{OA_1}, \ldots, \vec{OA_{i-1}}, \vec{OA_{i+1}}, \ldots, \vec{OA_{n+1}}$ is $1/n$ of that. It follows that the sum of the squares of the volumes of the $n+1$ non-hypotenuse faces is

$$\frac{1}{n^2}\sum_{j=1}^{n+1}\prod_{i=1,i\neq j}^{n+1} a_i^2$$

It must be shown that this is also the square of the volume of the hypotenuse face.

The hypotenuse face is determined by n vectors, but this can be done in more than one way. Here is one choice.

$$\begin{aligned}
\mathbf{v}_1 &= \vec{OA_1} - \vec{OA_2} = \langle a_1, -a_2, 0, \ldots, 0\rangle \\
\mathbf{v}_2 &= \vec{OA_1} - \vec{OA_3} = \langle a_1, 0, -a_3, \ldots, 0\rangle \\
&\vdots \\
\mathbf{v}_n &= \vec{OA_1} - \vec{OA_{n+1}} = \langle a_1, 0, 0, \ldots, -a_{n+1}\rangle
\end{aligned}$$

By Proposition 3.3, the volume of the hypotenuse face is $\frac{1}{n}$ times the magnitude of $\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle\!\rangle$. So what must be shown is the following.

$$|\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle\!\rangle|^2 = \sum_{j=1}^{n+1}\prod_{i=1,i\neq j}^{n+1} a_i^2 \tag{1}$$

The expansion of $\langle\!\langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle\!\rangle$ using Definition 3.1 has the form $X_1\mathbf{e}_1 + \cdots + X_{n+1}\mathbf{e}_{n+1}$ where each X_i is an $n \times n$ determinant. The claim is that determinant X_i evaluates to $\prod_{i=1,i\neq j}^{n+1} a_i$ (up to a factor of ± 1), which will give us the result.

The determinant X_1 has 0's above the main diagonal and $-a_2, -a_3, \ldots, -a_{n+1}$ along the main diagonal, so it evaluates to $(-1)^n(a_2a_3 \cdots a_{n+1})$. The other determinants are different than this, but similar to each other—as a representative case, take n to be 4, and consider X_4. Since exchanging two rows in a determinant changes its sign, we have the following.

$$X_4 = -\begin{vmatrix} a_1 & -a_2 & 0 & 0 \\ a_1 & 0 & -a_3 & 0 \\ a_1 & 0 & 0 & 0 \\ a_1 & 0 & 0 & -a_5 \end{vmatrix} = -\begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_1 & -a_2 & 0 & 0 \\ a_1 & 0 & -a_3 & 0 \\ a_1 & 0 & 0 & -a_5 \end{vmatrix} = a_1a_2a_3a_5$$

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