

A coordinate-free view on “the” generalized cross product

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Abstract

The higher dimensional generalization of the cross product is associated with an adequate matrix multiplication. This coordinate-free view allows for a better understanding of the underlying algebraic structures, among which are generalizations of Grassmann’s, Jacobi’s and Room’s identities. Moreover, such a view provides a the higher dimensional analogue of the decomposition of the vector Laplacian which itself gives an explicit coordinate-free Helmholtz decomposition in arbitrary dimensions $n \geq 2$.

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1 Introduction

The interplay between different differential operators is at the basis not only of pure analysis but also in many applied mathematical considerations. One possibility is to study instead of the properties of a linear homogeneous differential operator with constant coefficients

$$\mathcal{A} = \sum_{|\alpha|=k} A_{\alpha} \nabla^{\alpha} \tag{1.1a}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)^T \in \mathbb{N}_0^n$ is a multi-index of length $|\alpha| := \alpha_1 + \dots + \alpha_n$, $\nabla^{\alpha} := \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ and $A_{\alpha} \in \mathbb{R}^{N \times m}$, its symbol

$$\mathbb{A}(b) = \sum_{|\alpha|=k} A_{\alpha} b^{\alpha} \in \mathbb{R}^{N \times m}, \tag{1.1b}$$

where we used the notation $b^{\alpha} = b_1^{\alpha_1} \dots b_n^{\alpha_n}$ for $b \in \mathbb{R}^n$. Note that $\mathcal{A} : C_c^{\infty}(\Omega, \mathbb{R}^m) \rightarrow C_c^{\infty}(\Omega, \mathbb{R}^N)$ with $\Omega \subseteq \mathbb{R}^n$ open and we obtain for all $a \in C_c^{\infty}(\Omega, \mathbb{R}^m)$ also the expression $\mathcal{A}a = \mathcal{A}(\mathbb{D}a)$ with $\mathcal{A} \in \text{Lin}(\mathbb{R}^{m \times n}, \mathbb{R}^N)$. The approach to look and algebraically operate with the vector differential operator ∇ in a manner of a vector is also referred as vector calculus or formal calculations.

An example of such differential operator is the derivative \mathbb{D} itself, but also div , curl , Δ or **inc**. One of the most prominent relation in vector calculus is $\text{curl} \nabla \zeta \equiv 0$ for scalar fields $\zeta \in C_c^{\infty}(\Omega)$, $\Omega \subseteq \mathbb{R}^3$ open, which from an algebraic point of view reads $b \times b = 0$ for all $b \in \mathbb{R}^3$ (where a scalar factor can be and was omitted).

Here, we focus on a higher dimensional analogue of the curl or rather on the study of the underlying generalized cross product. An extension of the usual cross product of vectors in \mathbb{R}^3 to vectors in \mathbb{R}^n depends on the properties one requires to hold. The three basic properties of the vector product are: the bilinearity in both arguments, that the vector $a \times b$ is perpendicular to both $a, b \in \mathbb{R}^3$ (and, thus belongs to

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the same space) and that its length is the area of the parallelogram spanned by a and b . Gibbs uses these properties also to define the cross product, see [6, Chapter II]. It turns out, that such a vector product exists only in three and seven dimensions, cf. [17]. However, the 7-dimensional vector product does not satisfy Jacobi's identity but rather a generalization of it, namely the *Malcev identity*, cf. [3, p. 279] and the references contained at the end of the section therein. We will not follow those constructions here and will generalize the cross product to all dimensions dropping one of its basic properties instead. These considerations are usually done using coordinates. However, we will dwell on its coordinate-free view which offers a better understanding of the underlying algebraic structures. Such a view already turned out to be very fruitful in extending Korn inequalities for incompatible tensor fields to higher dimensions, cf. [15] where first thoughts in that matrix representations have been investigated. However, we will catch up here with the underlying algebraic structures, among which are generalizations of Grassmann's, Jacobi's and Room's identities. Moreover, such a view provides a the higher dimensional analogue of the decomposition of the vector Laplacian which itself gives an explicit coordinate-free Helmholtz decomposition in arbitrary dimensions $n \geq 2$.

2 Notations

As usual $\cdot \otimes \cdot$ and $\langle \cdot, \cdot \rangle$ denote the dyadic and the scalar product, respectively. The space of symmetric $(n \times n)$ -matrices will be denoted by $\text{Sym}(n)$ and the space of skew-symmetric $(n \times n)$ -matrices by $\mathfrak{so}(n)$. We will use lower-case Greek letters to denote scalars, lower-case Latin letters to denote column vectors and upper-case Latin letters to denote matrices, with two exceptions for the dimensions: if not otherwise stated we have $n, m, N \in \mathbb{N}$ and $n \geq 2$. The identity matrix will be denoted by I_n . $\text{sym } P$, skew P and P^T denote the symmetric part, the skew-symmetric part and the transpose of a matrix P , respectively.

3 Algebraic view of “the” generalized cross product

3.1 Inductive introduction

From an algebraic point of view the components of the cross product $a \times b$ are of the form $\alpha_i \beta_j - \alpha_j \beta_i$ for $1 \leq i < j \leq 3$ sorted (and multiplied with -1) in such a way that the resulting vector is perpendicular to both a and b . For a general $n \in \mathbb{N}$ we have $\frac{n(n-1)}{2}$ combinations of the form $\alpha_i \beta_j - \alpha_j \beta_i$ with $1 \leq i < j \leq n$ and only for $n = 3$ this number corresponds with the space dimension. So, we will drop the orthogonality condition and consider a generalized cross product \times_n in all dimensions $n \geq 2$ which is anti-commutative, bilinear and whose length is the area of the parallelogram spanned by a and b (see (3.9) below):

$$a \times_n b = \begin{pmatrix} \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ \alpha_1 \beta_3 - \alpha_3 \beta_1 \\ \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_1 \beta_4 - \alpha_4 \beta_1 \\ \alpha_2 \beta_4 - \alpha_4 \beta_2 \\ \alpha_3 \beta_4 - \alpha_4 \beta_3 \\ \vdots \end{pmatrix} \quad \text{for } a = (\alpha_i)_{i=1, \dots, n}, b = (\beta_i)_{i=1, \dots, n} \in \mathbb{R}^n. \quad (3.1)$$

Thus, using the following notation

$$b = (\bar{b}, \beta_n)^T \in \mathbb{R}^n \quad \text{with } \bar{b} \in \mathbb{R}^{n-1} \quad (3.2)$$

the generalized cross product \times_n is inductively given by

$$a \times_n b := \begin{pmatrix} \bar{a} \times_{n-1} \bar{b} \\ \beta_n \cdot \bar{a} - \alpha_n \cdot \bar{b} \end{pmatrix} \in \mathbb{R}^{\frac{n(n-1)}{2}} \quad \text{where} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \times_2 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} := \alpha_1 \beta_2 - \alpha_2 \beta_1, \quad (3.3)$$

wherefrom the bilinearity and anti-commutativity follow immediately.

3.2 Relation to skew-symmetric matrices

To establish the connection of the generalized cross product $a \times_n b$ to the entries of $\text{skew}(a \otimes b)$ we start with the following bijection $\mathfrak{a}_n : \mathfrak{so}(n) \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$ given by

$$\mathfrak{a}_n(A) := (\alpha_{12}, \alpha_{13}, \alpha_{23}, \dots, \alpha_{1n}, \dots, \alpha_{(n-1)n})^T \quad (3.4a)$$

for $A = (\alpha_{ij})_{i,j=1,\dots,n} \in \mathfrak{so}(n)$, as well as its inverse $\mathfrak{A}_n : \mathbb{R}^{\frac{n(n-1)}{2}} \rightarrow \mathfrak{so}(n)$, so that

$$\begin{aligned} \mathfrak{A}_n(\mathfrak{a}_n(A)) &= A \quad \forall A \in \mathfrak{so}(n) \quad \text{and} \\ \mathfrak{a}_n(\mathfrak{A}_n(a)) &= a \quad \forall a \in \mathbb{R}^{\frac{n(n-1)}{2}} \end{aligned} \quad (3.4b)$$

and for $a = (\alpha_1, \dots, \alpha_{\frac{n(n-1)}{2}})^T \in \mathbb{R}^{\frac{n(n-1)}{2}}$ in coordinates it looks like

$$\mathfrak{A}_n(a) = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_4 & \vdots \\ -\alpha_1 & 0 & \alpha_3 & \alpha_5 & \vdots \\ -\alpha_2 & -\alpha_3 & 0 & \alpha_6 & \vdots \\ -\alpha_4 & -\alpha_5 & -\alpha_6 & 0 & \vdots \\ \dots & \dots & \dots & \dots & 0 \end{pmatrix}. \quad (3.5)$$

Thus, the generalized cross product $a \times_n b$ can be written as

$$a \times_n b = \mathfrak{a}_n(a \otimes b - b \otimes a), \quad (3.6a)$$

or, equivalently, it holds

$$\mathfrak{A}_n(a \times_n b) = a \otimes b - b \otimes a \quad \text{for } a, b \in \mathbb{R}^n. \quad (3.6b)$$

3.3 Lagrange's identity

However, the inductive definition (3.3) can be used to directly deduce an analogue to *Lagrange's identity*:

$$\langle a \times_n b, c \times_n d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle \quad \forall a, b, c, d \in \mathbb{R}^n. \quad (3.7)$$

Indeed, in $n = 2$ dimensions we have

$$\begin{aligned} \left\langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \times_2 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \times_2 \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \right\rangle &= (\alpha_1 \beta_2 - \alpha_2 \beta_1)(\gamma_1 \delta_2 - \gamma_2 \delta_1) \\ &= \alpha_1 \beta_2 \gamma_1 \delta_2 + \alpha_2 \beta_1 \gamma_2 \delta_1 - \alpha_1 \beta_2 \gamma_2 \delta_1 - \alpha_2 \beta_1 \gamma_1 \delta_2 \\ &= (\alpha_1 \gamma_1 + \alpha_2 \gamma_2)(\beta_1 \delta_1 + \beta_2 \delta_2) - (\alpha_1 \delta_1 + \alpha_2 \delta_2)(\beta_1 \gamma_1 + \beta_2 \gamma_2) \\ &= \left\langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right\rangle. \end{aligned}$$

Furthermore, with $a = (\bar{a}, \alpha_n)^T, b = (\bar{b}, \beta_n)^T, c = (\bar{c}, \gamma_n)^T, d = (\bar{d}, \delta_n)^T$ we obtain on one hand

$$\begin{aligned} \langle a \times_n b, c \times_n d \rangle &= \left\langle \begin{pmatrix} \bar{a} \times_{n-1} \bar{b} \\ \beta_n \cdot \bar{a} - \alpha_n \cdot \bar{b} \end{pmatrix}, \begin{pmatrix} \bar{c} \times_{n-1} \bar{d} \\ \delta_n \cdot \bar{c} - \gamma_n \cdot \bar{d} \end{pmatrix} \right\rangle \\ &= \langle \bar{a} \times_{n-1} \bar{b}, \bar{c} \times_{n-1} \bar{d} \rangle + \beta_n \delta_n \langle \bar{a}, \bar{c} \rangle + \alpha_n \gamma_n \langle \bar{b}, \bar{d} \rangle - \beta_n \gamma_n \langle \bar{a}, \bar{d} \rangle - \alpha_n \delta_n \langle \bar{b}, \bar{c} \rangle, \end{aligned}$$

and on the other hand:

$$\begin{aligned}
\langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle &= \\
&= (\langle \bar{a}, \bar{c} \rangle + \alpha_n \gamma_n) (\langle \bar{b}, \bar{d} \rangle + \beta_n \delta_n) - (\langle \bar{a}, \bar{d} \rangle + \alpha_n \delta_n) (\langle \bar{b}, \bar{c} \rangle + \beta_n \gamma_n) \\
&= \langle \bar{a}, \bar{c} \rangle \langle \bar{b}, \bar{d} \rangle - \langle \bar{a}, \bar{d} \rangle \langle \bar{b}, \bar{c} \rangle + \beta_n \delta_n \langle \bar{a}, \bar{c} \rangle + \alpha_n \gamma_n \langle \bar{b}, \bar{d} \rangle - \beta_n \gamma_n \langle \bar{a}, \bar{d} \rangle - \alpha_n \delta_n \langle \bar{b}, \bar{c} \rangle,
\end{aligned} \tag{3.8}$$

so that (3.7) follows by induction over $n \in \mathbb{N}$, $n \geq 2$. Especially, for $c = a$ and $d = b$ we obtain for the squared norm of the generalized cross product

$$\|a \times_n b\|^2 \stackrel{(3.7)}{=} \|a\|^2 \|b\|^2 - \langle a, b \rangle^2 \quad \forall a, b \in \mathbb{R}^n \tag{3.9}$$

meaning that the length of $a \times_n b$ is equal to the area of the parallelogram spanned by the vectors a and b .

Two (non-zero) vectors $a, b \in \mathbb{R}^n$ are linearly dependent (and thus *parallel*) if and only if $a \times_n b = 0$.

3.4 Matrix representation

It is well known, that an identification of the usual cross product \times with an adequate matrix multiplication facilitated some of the common proofs in vector algebra and allowed to extend the cross product of vectors to a cross product of a vector and a matrix, cf. [19, 8, 21, 13]. It will be our next goal to achieve a similar identification of the generalized cross product \times_n with a corresponding matrix multiplication. Indeed, since for a fixed $a \in \mathbb{R}^n$ the operation $a \times_n \cdot$ is linear in the second component there exists a unique matrix denoted by $\llbracket a \rrbracket_{\times_n} \in \mathbb{R}^{\frac{n(n-1)}{2} \times n}$ such that

$$a \times_n b =: \llbracket a \rrbracket_{\times_n} b \quad \forall b \in \mathbb{R}^n. \tag{3.10}$$

In view of (3.3) the matrices $\llbracket \cdot \rrbracket_{\times_n}$ can be characterized inductively, and for $a = (\bar{a}, \alpha_n)^T$ the matrix $\llbracket a \rrbracket_{\times_n}$ has the form

$$\llbracket a \rrbracket_{\times_n} = \begin{pmatrix} \llbracket \bar{a} \rrbracket_{\times_{n-1}} & \vdots & 0 \\ \dots & \dots & \dots \\ -\alpha_n \cdot I_{n-1} & \vdots & \bar{a} \end{pmatrix} \quad \text{where} \quad \llbracket \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rrbracket_{\times_2} = \begin{pmatrix} -\alpha_2 & \alpha_1 \end{pmatrix}, \tag{3.11}$$

so that

$$\llbracket \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \rrbracket_{\times_3} = \begin{pmatrix} -\alpha_2 & \alpha_1 & \vdots & 0 \\ \dots & \dots & \dots & \dots \\ -\alpha_3 & 0 & \vdots & \alpha_1 \\ 0 & -\alpha_3 & \vdots & \alpha_2 \end{pmatrix} \quad \text{etc.} \tag{3.12}$$

Remark 3.1. The entries of the generalized cross product $a \times_3 b$, with $a, b \in \mathbb{R}^3$, are permutations (with a sign) of the entries of the classical cross product $a \times b$. Recall, that the operation $a \times \cdot$ can be identified with a multiplication with the skew-symmetric matrix

$$\text{Anti}(a) = \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix} \tag{3.13}$$

which differs from the expression $\llbracket a \rrbracket_{\times_3}$ for $a = (\alpha_1, \alpha_2, \alpha_3)^T$ and also from $\mathfrak{A}_3(a)$ which reads

$$\mathfrak{A}_3(a) = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_1 & 0 & \alpha_3 \\ -\alpha_2 & -\alpha_3 & 0 \end{pmatrix}. \tag{3.14}$$

Indeed, also the notations T_a , $W(a)$ or even $[a]_{\times}$ are used for $\text{Anti}(a)$, however, the latter emphasizes that we deal with a skew-symmetric matrix.

Remark 3.2. Also the 7-dimensional vector product $a \times \cdot$ for $a \in \mathbb{R}^7$ (which differs from $a \times_7 \cdot$) can be represented by a multiplication with a skew-symmetric matrix from $\mathfrak{so}(7)$, see [11, 12, 2].

3.5 Scalar triple product

Since $\times_n : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$ it does not make sense to think of an analogue of a scalar triple product with three vectors coming from the same vector space but rather instead:

$$\langle \mathbf{a}, b \times_n c \rangle = \langle \mathbf{a}, \llbracket b \rrbracket_{\times_n} c \rangle = \langle \llbracket b \rrbracket_{\times_n}^T \mathbf{a}, c \rangle \quad \forall \mathbf{a} \in \mathbb{R}^{\frac{n(n-1)}{2}}, b, c \in \mathbb{R}^n, \quad (3.15a)$$

so that with $c = b$ we have:

$$\langle \llbracket b \rrbracket_{\times_n}^T \mathbf{a}, b \rangle = 0 \quad \forall \mathbf{a} \in \mathbb{R}^{\frac{n(n-1)}{2}}, b \in \mathbb{R}^n. \quad (3.15b)$$

Note the slight difference to the case of the usual cross product. The latter can be represented by through a multiplication with a square skew-symmetric matrix whereas the generalized cross product through matrices of the form (3.11) whom are wether square matrices (except the case $n = 3$) nor skew-symmetric. So, in the case of the usual cross product we regain after swapping the matrix in the triple scalar product again the considered matrix (with a changed sign):

$$\langle a, b \times c \rangle = \langle a, \text{Anti}(b) c \rangle = -\langle \text{Anti}(b) a, c \rangle = \langle a \times b, c \rangle, \quad a, b, c \in \mathbb{R}^3. \quad (3.16)$$

On the contrary, in the case of the generalized cross product, we have to deal with matrices $\llbracket \cdot \rrbracket_{\times_n}$, see (3.15). Such matrices will be very important in the subsequent considerations.

3.6 Grassmann's identity

Also in a generalization of a vector triple product we cannot consider the double appearance of the generalized cross product but as in the generalization of the scalar triple focus on the matrix $\llbracket \cdot \rrbracket_{\times_n}$. Thus, as a generalization of *Grassmann's identity* we obtain for $a, b, c \in \mathbb{R}^n$ instead:

$$\begin{aligned} \llbracket a \rrbracket_{\times_n}^T (b \times_n c) &= \langle a, b \rangle \cdot c - \langle a, c \rangle \cdot b = (\langle b, a \rangle \cdot I_n - b \otimes a) c \\ &= (c \otimes b - b \otimes c) a \stackrel{(3.6)}{=} -\mathfrak{A}_n(b \times_n c) a \in \mathbb{R}^n. \end{aligned} \quad (3.17)$$

Indeed, to establish the first equality (3.17)₁ in $n = 2$ dimensions we consider

$$\begin{aligned} \begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix} \left(\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \times_2 \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) &= \begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix} \cdot (\beta_1 \gamma_2 - \beta_2 \gamma_1) = \begin{pmatrix} \alpha_2 \beta_2 \gamma_1 - \alpha_2 \beta_1 \gamma_2 \\ \alpha_1 \beta_1 \gamma_2 - \alpha_1 \beta_2 \gamma_1 \end{pmatrix} \\ &= \left\langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right\rangle \cdot \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} - \left\langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right\rangle \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}. \end{aligned} \quad (3.18)$$

Furthermore, with $a = (\bar{a}, \alpha_n)^T, b = (\bar{b}, \beta_n)^T, c = (\bar{c}, \gamma_n)^T$ we obtain on one hand

$$\begin{aligned} \llbracket a \rrbracket_{\times_n}^T (b \times_n c) &= \begin{pmatrix} \llbracket \bar{a} \rrbracket_{\times_{n-1}}^T & \vdots & -\alpha_n \cdot I_{n-1} \\ \dots & \vdots & \dots \\ 0 & \vdots & \bar{a}^T \end{pmatrix} \begin{pmatrix} \bar{b} \times_{n-1} \bar{c} \\ \gamma_n \cdot \bar{b} - \beta_n \cdot \bar{c} \end{pmatrix} \\ &= \begin{pmatrix} \llbracket \bar{a} \rrbracket_{\times_{n-1}}^T (\bar{b} \times_{n-1} \bar{c}) + \alpha_n \beta_n \cdot \bar{c} - \alpha_n \gamma_n \cdot \bar{b} \\ \gamma_n \langle \bar{a}, \bar{b} \rangle - \beta_n \langle \bar{a}, \bar{c} \rangle \end{pmatrix} \end{aligned} \quad (3.19)$$

and on the other hand:

$$\begin{aligned} \langle a, b \rangle \cdot c - \langle a, c \rangle \cdot b &= \left\langle \begin{pmatrix} \bar{a} \\ \alpha_n \end{pmatrix}, \begin{pmatrix} \bar{b} \\ \beta_n \end{pmatrix} \right\rangle \cdot \begin{pmatrix} \bar{c} \\ \gamma_n \end{pmatrix} - \left\langle \begin{pmatrix} \bar{a} \\ \alpha_n \end{pmatrix}, \begin{pmatrix} \bar{c} \\ \gamma_n \end{pmatrix} \right\rangle \cdot \begin{pmatrix} \bar{b} \\ \beta_n \end{pmatrix} \\ &= \begin{pmatrix} \langle \bar{a}, \bar{b} \rangle \cdot \bar{c} - \langle \bar{a}, \bar{c} \rangle \cdot \bar{b} + \alpha_n \beta_n \cdot \bar{c} - \alpha_n \gamma_n \cdot \bar{b} \\ \langle \bar{a}, \bar{b} \rangle \gamma_n - \langle \bar{a}, \bar{c} \rangle \beta_n \end{pmatrix} \end{aligned} \quad (3.20)$$

so that (3.17)₁ follows by induction over $n \in \mathbb{N}$, $n \geq 2$.

3.7 Jacobi's identity

We obtain the following generalization of *Jacobi's identity*:

$$\llbracket a \rrbracket_{\times_n}^T (b \times_n c) + \llbracket b \rrbracket_{\times_n}^T (c \times_n a) + \llbracket c \rrbracket_{\times_n}^T (a \times_n b) \stackrel{(3.17)_1}{=} 0 \quad \forall a, b, c \in \mathbb{R}^n \quad (3.21a)$$

or, equivalently:

$$\mathfrak{A}_n(b \times_n c) a + \mathfrak{A}_n(c \times_n a) b + \mathfrak{A}_n(a \times_n b) c \stackrel{(3.17)_4}{=} 0. \quad (3.21b)$$

Surely, relation (3.17) can also be used to obtain (3.7).

3.8 Cross product with a matrix

Furthermore, the generalized cross product can be written as

$$a \times_n b = -b \times_n a = \llbracket -b \rrbracket_{\times_n} a = \left(a^T \llbracket -b \rrbracket_{\times_n}^T \right)^T \quad (3.22)$$

this allows us to define a generalized cross product of a vector $b \in \mathbb{R}^n$ and a matrix $P \in \mathbb{R}^{m \times n}$ from the right and with a matrix $B \in \mathbb{R}^{n \times m}$ from the left, where $m \in \mathbb{N}$, via

$$P \times_n b := P \llbracket -b \rrbracket_{\times_n}^T \in \mathbb{R}^{m \times \frac{n(n-1)}{2}} \quad \text{seen as row-wise cross product,} \quad (3.23a)$$

and

$$b \times_n B := \llbracket b \rrbracket_{\times_n} B \in \mathbb{R}^{\frac{n(n-1)}{2} \times m} \quad \text{seen as column-wise cross product,} \quad (3.23b)$$

and they are connected via

$$(b \times_n B)^T = B^T \llbracket b \rrbracket_{\times_n}^T = -B^T \times_n b \quad \forall B \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n. \quad (3.23c)$$

So, especially for the identity matrix $P = I_n$ we obtain

$$I_n \times_n b = \llbracket -b \rrbracket_{\times_n}^T \quad \text{and} \quad b \times_n I_n = \llbracket b \rrbracket_{\times_n}. \quad (3.24)$$

Moreover, for $a \in \mathbb{R}^m$ and $b, c \in \mathbb{R}^n$ it follows

$$(a \otimes b) \times_n c = a b^T \llbracket -c \rrbracket_{\times_n}^T = a (\llbracket -c \rrbracket_{\times_n} b)^T = a (-c \times_n b)^T = a \otimes (b \times_n c), \quad (3.25a)$$

and, especially, for $c = b$:

$$(a \otimes b) \times_n b = 0 \quad \text{for all } a \in \mathbb{R}^m \text{ and all } b \in \mathbb{R}^n. \quad (3.25b)$$

As consequence we obtain for $a, b \in \mathbb{R}^n$:

$$\begin{aligned} (b \otimes a) \times_n b &\stackrel{(3.25b)}{=} 2 \cdot \text{sym}(a \otimes b) \times_n b = -2 \cdot \text{skew}(a \otimes b) \times_n b \\ &\stackrel{(3.25a)}{=} b \otimes (a \times_n b) \stackrel{(3.6)}{=} 2 \cdot b \otimes \mathfrak{a}_n(\text{skew}(a \otimes b)). \end{aligned} \quad (3.25c)$$

3.9 Another vector triple

Already in the scalar triple product we came across with the expression $\llbracket b \rrbracket_{\times_n}^T \mathbf{a} \in \mathbb{R}^n$. Hence, we may consider also the following vector triple product for $\mathbf{a} \in \mathbb{R}^{\frac{n(n-1)}{2}}$ and $b, c \in \mathbb{R}^n$:

$$\begin{aligned} \left(\llbracket b \rrbracket_{\times_n}^T \mathbf{a} \right) \times_n c &= \left[\llbracket b \rrbracket_{\times_n}^T \mathbf{a} \right]_{\times_n} c \\ &= -c \times_n \llbracket b \rrbracket_{\times_n}^T \mathbf{a} = -\llbracket c \rrbracket_{\times_n} \llbracket b \rrbracket_{\times_n}^T \mathbf{a} = (\llbracket c \rrbracket_{\times_n} \times_n b) \mathbf{a} \in \mathbb{R}^{\frac{n(n-1)}{2}}. \end{aligned} \quad (3.26)$$

Again, the corresponding relations to (3.17) and (3.26) for the usual cross product coincide where the situation is different for the generalized cross product due to the non-symmetry of the corresponding matrices. An inductive view on the appearing matrix in (3.26) shows for all $a, b \in \mathbb{R}^n$:

$$\begin{aligned} \llbracket a \rrbracket_{\times_n} \times_n b &= \llbracket a \rrbracket_{\times_n} \llbracket -b \rrbracket_{\times_n}^T = \\ &= \begin{pmatrix} \llbracket \bar{a} \rrbracket_{\times_{n-1}} \times_{n-1} \bar{b} & \vdots & \beta_n \cdot \llbracket \bar{a} \rrbracket_{\times_{n-1}} \\ \dots & \dots & \dots \\ \alpha_n \cdot \llbracket \bar{b} \rrbracket_{\times_{n-1}}^T & \vdots & -\bar{a} \otimes \bar{b} - \alpha_n \beta_n \cdot I_{n-1} \end{pmatrix} \in \mathbb{R}^{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}}, \end{aligned} \quad (3.27)$$

and, especially, for $a = b$:

$$\llbracket b \rrbracket_{\times_n} \times_n b = -\llbracket b \rrbracket_{\times_n} \llbracket b \rrbracket_{\times_n}^T \in \text{Sym}\left(\frac{n(n-1)}{2}\right). \quad (3.28)$$

Consequently, we may also consider the following matrix multiplication:

$$\mathfrak{P} \llbracket b \rrbracket_{\times_n} \in \mathbb{R}^{m \times n} \quad \text{for } \mathfrak{P} \in \mathbb{R}^{m \times \frac{n(n-1)}{2}} \quad (3.29)$$

and, like in (3.23), related by transposition also $\llbracket b \rrbracket_{\times_n}^T (\cdot)$ for an $\left(\frac{n(n-1)}{2} \times m\right)$ -matrix.

3.10 Room's identity

Surely, the considerations in the previous subsections were inspired by the corresponding relations known for the usual cross product. So, from the usual Grassmann's identity one can deduce the usual Jacobi's and Lagrange's identities. Moreover, the usual Grassmann's identity for the vector triple allows also to conclude

$$\text{Anti}(a) \text{Anti}(b) = \text{Anti}(a) \times b = b \otimes a - \langle a, b \rangle \cdot I_3 \quad \forall a, b \in \mathbb{R}^3. \quad (3.30)$$

This algebraic relation is already contained in [19, p. 691 (ii)]. For that reason let us call it *Room's identity*. This relation (3.30) turned out to be very important also from an application point of view, cf. [13, 14] and the references contained therein.

Returning to the n -dimensional case, we have for arbitrary $a, b \in \mathbb{R}^n$:

$$\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n} x = \llbracket a \rrbracket_{\times_n}^T (b \times_n x) \stackrel{(3.17)}{=} (\langle b, a \rangle \cdot I_n - b \otimes a) x \quad \forall x \in \mathbb{R}^n, \quad (3.31)$$

so that as an analogue to *Room's identity* it follows

$$\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n} = \langle b, a \rangle \cdot I_n - b \otimes a \in \mathbb{R}^{n \times n} \quad \forall a, b \in \mathbb{R}^n, \quad (3.32)$$

and, especially, for $a = b$:

$$\llbracket b \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n} = \|b\|^2 \cdot I_n - b \otimes b \in \text{Sym}(n). \quad (3.33)$$

Interchanging the roles of a and b in (3.32) we further deduce

$$\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n} - \llbracket b \rrbracket_{\times_n}^T \llbracket a \rrbracket_{\times_n} = a \otimes b - b \otimes a \stackrel{(3.6)}{=} \mathfrak{A}_n(a \times_n b). \quad (3.34)$$

Since $\text{tr}(a \otimes b) = \langle a, b \rangle$, the expression (3.32) shows, that the entries of $\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n}$ are linear combinations of the entries of the dyadic product $a \otimes b$. Again, also the converse holds true:

$$b \otimes a = \frac{\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n}}{n-1} \cdot I_n - \llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n}, \quad (3.35)$$

where we leave it as an exercise for the reader to verify (e.g., by induction) that

$$\text{tr}(\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n}) = \langle \llbracket a \rrbracket_{\times_n}, \llbracket b \rrbracket_{\times_n} \rangle = (n-1) \langle a, b \rangle. \quad (3.36)$$

Recall, that the associated matrix $\text{Anti}(\cdot)$ with the usual cross product \times in \mathbb{R}^3 is a (skew-symmetric) square matrix, whereas the associated matrix $\llbracket \cdot \rrbracket_{\times_n}$ with the generalized cross product \times_n is an $(\frac{n(n-1)}{2} \times n)$ -matrix and, thus, only for $n = 3$ a square matrix. Hence, despite of the situation in Room's identity (3.30) we may also interchange the matrices in its n -dimensional analogue (3.32), i.e., consider the expression in (3.26).

Returning to the usual *Room's identity* we have

$$\text{Anti}(a) \times b = L(a \otimes b) \quad \text{and} \quad a \otimes b = L(\text{Anti}(a) \times b) \quad \forall a, b \in \mathbb{R}^3. \quad (3.37a)$$

On one hand, we associate with the matrix $\text{Anti}(\cdot)$ a representation of the cross product. *Room's identity* can be generalized to higher dimensions in three different ways. We have already seen in (3.32) and (3.35) an extension to:

$$\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n} = L(a \otimes b) \quad \text{and} \quad a \otimes b = L(\llbracket a \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n}) \quad \forall a, b \in \mathbb{R}^n. \quad (3.37b)$$

However, a similar result to (3.37a) also holds true for the generalized cross product of the matrix coming from the matrix representation of the generalized cross product with a vector, see [15]:

$$\begin{aligned} \llbracket a \rrbracket_{\times_n} \times_n b &= L(a \otimes b) \quad \forall a, b \in \mathbb{R}^n, n \geq 2 \\ \text{and} \quad a \otimes b &= L(\llbracket a \rrbracket_{\times_n} \times_n b) \quad \forall a, b \in \mathbb{R}^n, n \geq 3. \end{aligned} \quad (3.37c)$$

These cover also the case of $a \times_n \llbracket b \rrbracket_{\times_n}^T = \llbracket a \rrbracket_{\times_n} \llbracket b \rrbracket_{\times_n}^T = -\llbracket a \rrbracket_{\times_n} \times_n b$, which for $n = 2$ is just a scalar.

On the other hand, *Room's identity* can also be seen as an expression for the cross product of a skew-symmetric matrix with a vector:

$$A \times b = L(\text{axl}(A) \otimes b) \quad \text{and} \quad \text{axl}(A) \otimes b = L(A \times b) \quad \forall A \in \mathfrak{so}(3), b \in \mathbb{R}^3, \quad (3.37a')$$

where $\text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ denotes the inverse of $\text{Anti}(\cdot)$. Interestingly, a similar result holds true for $(n \times n)$ -skew symmetric matrices in all dimensions $n \geq 2$, see [15]:

$$\begin{aligned} A \times_n b &= L(\mathfrak{a}_n(A) \otimes b) \\ \text{and} \quad \mathfrak{a}_n(A) \otimes b &= L(A \times_n b) \quad \forall A \in \mathfrak{so}(n), b \in \mathbb{R}^n, \end{aligned} \quad (3.37d)$$

where (3.37c)₁ and (3.37d)₁ follow directly from the definition of the generalized cross product of a matrix and a vector but for (3.37c)₂ and (3.37d)₂ inductive proofs are needed, cf. [15].

Remark 3.3. We have seen, that *Room's identity* (3.30) admits three different generalizations to higher dimensions (3.37b), (3.37c), (3.37d) which coincide when considering the usual cross product and the associated matrix to it since the latter is a skew-symmetric (square) matrix. However, *Grassmann's* and *Jacobi's* identities generalize only in the ways presented in (3.17) and (3.21) which are comparable to the situation considering the usual triple vector product $a \times (b \times c) = \text{Anti}(a) (b \times c)$ since $\text{Anti}(a)^T = -\text{Anti}(a)$.

3.11 Simultaneous cross product

Of special interest is a simultaneous cross product of a square matrix $P \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$ from both sides:

$$b \times_n P \times_n b = \llbracket b \rrbracket_{\times_n} P \llbracket -b \rrbracket_{\times_n}^T \stackrel{(3.23c)}{=} -((P \times_n b)^T \times_n b)^T, \quad (3.38)$$

where, due to the associativity of matrix multiplication, we can omit parenthesis. Since

$$(b \times_n P \times_n b)^T \stackrel{(3.38)}{=} -\llbracket b \rrbracket_{\times_n} P^T \llbracket -b \rrbracket_{\times_n}^T = b \times_n P^T \times_n b \quad (3.39a)$$

it follows for $S \in \text{Sym}(n)$ and $A \in \mathfrak{so}(n)$:

$$b \times_n S \times_n b \in \text{Sym}\left(\frac{n(n-1)}{2}\right) \quad \text{and} \quad b \times_n A \times_n b \in \mathfrak{so}\left(\frac{n(n-1)}{2}\right) \quad (3.39b)$$

but also for all $P \in \mathbb{R}^{n \times n}$:

$$b \times_n \text{sym} P \times_n b = \text{sym}(b \times_n P \times_n b), \quad b \times_n \text{skew} P \times_n b = \text{skew}(b \times_n P \times_n b). \quad (3.39c)$$

For $P = I_n$ the identity matrix we obtain

$$b \times_n I_n \times_n b = \llbracket b \rrbracket_{\times_n} \llbracket -b \rrbracket_{\times_n}^T \stackrel{(3.28)}{=} \llbracket b \rrbracket_{\times_n} \times_n b \in \text{Sym}\left(\frac{n(n-1)}{2}\right). \quad (3.40)$$

Moreover, for $a, b, c \in \mathbb{R}^n$ it follows

$$b \times_n (a \otimes c) \times_n b \stackrel{(3.25a)}{=} (b \times_n a) \otimes (c \times_n b), \quad (3.41a)$$

and, especially, for $c = b$

$$\begin{aligned} b \times_n (a \otimes b) \times_n b &= b \times_n (b \otimes a) \times_n b \\ &= b \times_n \text{sym}(a \otimes b) \times_n b = b \times_n \text{skew}(a \otimes b) \times_n b = 0. \end{aligned} \quad (3.41b)$$

Furthermore, for a square matrix $\mathfrak{P} \in \mathbb{R}^{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}}$ and a vector $b \in \mathbb{R}^n$ we obtain

$$\llbracket b \rrbracket_{\times_n}^T \mathfrak{P} \llbracket b \rrbracket_{\times_n} \in \mathbb{R}^{n \times n}, \quad (3.42)$$

which has comparable properties to the simultaneous cross product above, for instance:

$$\left(\llbracket b \rrbracket_{\times_n}^T \mathfrak{P} \llbracket b \rrbracket_{\times_n} \right)^T = \llbracket b \rrbracket_{\times_n}^T \mathfrak{P}^T \llbracket b \rrbracket_{\times_n} \quad (3.43a)$$

which gives:

$$\text{sym} \left(\llbracket b \rrbracket_{\times_n}^T \mathfrak{P} \llbracket b \rrbracket_{\times_n} \right) = \llbracket b \rrbracket_{\times_n}^T \text{sym} \mathfrak{P} \llbracket b \rrbracket_{\times_n}, \quad (3.43b)$$

as well as

$$\text{skew} \left(\llbracket b \rrbracket_{\times_n}^T \mathfrak{P} \llbracket b \rrbracket_{\times_n} \right) = \llbracket b \rrbracket_{\times_n}^T \text{skew} \mathfrak{P} \llbracket b \rrbracket_{\times_n}. \quad (3.43c)$$

And for the identity matrix $\mathfrak{P} = I_{\frac{n(n-1)}{2}}$ we obtain:

$$\llbracket b \rrbracket_{\times_n}^T I_{\frac{n(n-1)}{2}} \llbracket b \rrbracket_{\times_n} = \llbracket b \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n} \stackrel{(3.33)}{=} \|b\|^2 \cdot I_n - b \otimes b. \quad (3.44)$$

Again, the corresponding expressions to (3.38) and (3.42) coming from the usual cross product just coincide.

4 Differential operators

Let us now come back to the interplay between a linear homogeneous differential operator with constant coefficients and its symbol, thus, replacing b by the vector differential operator ∇ in the algebraic relation presented in the previous section. For that purpose let $\Omega \subseteq \mathbb{R}^n$ be open, $n \geq 2$ and $n, m \in \mathbb{N}$. As usual, the derivative and the divergence of a vector field rely on the dyadic product and the scalar product, respectively:

$$\begin{aligned} D a &:= a \otimes \nabla \in C_c^\infty(\Omega, \mathbb{R}^{m \times n}) \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^m), \\ \operatorname{div} a &:= \langle a, \nabla \rangle = a^T \nabla = \operatorname{tr}(D a) \in C_c^\infty(\Omega, \mathbb{R}) \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^n), \end{aligned} \quad (4.1)$$

where the latter generalizes to a matrix divergence which is taken row-wise:

$$\operatorname{Div} P := P \nabla \in C_c^\infty(\Omega, \mathbb{R}^m) \quad \text{for } P \in C_c^\infty(\Omega, \mathbb{R}^{m \times n}). \quad (4.2)$$

Similarly, the generalized curl is related to the generalized cross product via

$$\begin{aligned} \operatorname{curl}_n a &:= a \times_n (-\nabla) = \nabla \times_n a = \llbracket \nabla \rrbracket_{\times_n} a \\ &\stackrel{(3.6)}{=} \mathbf{a}_n(\operatorname{skew} D a) \in C_c^\infty(\Omega, \mathbb{R}^{\frac{n(n-1)}{2}}) \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^n), \end{aligned} \quad (4.3)$$

where the latter expression which is usually considered in coordinates to introduce the generalized curl.

Furthermore, we consider the new differential operation

$$\llbracket \nabla \rrbracket_{\times_n}^T \mathbf{a} \in C_c^\infty(\Omega, \mathbb{R}^n) \quad \text{for } \mathbf{a} \in C_c^\infty(\Omega, \mathbb{R}^{\frac{n(n-1)}{2}}), \quad (4.4)$$

which differs from $\operatorname{curl}_{\frac{n(n-1)}{2}} \mathbf{a}$ also in the three-dimensional case:

$$\operatorname{curl}_3 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \partial_1 \alpha_2 - \partial_2 \alpha_1 \\ \partial_1 \alpha_3 - \partial_3 \alpha_1 \\ \partial_2 \alpha_3 - \partial_3 \alpha_2 \end{pmatrix}, \quad \llbracket \nabla \rrbracket_{\times_3}^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -\partial_2 \alpha_1 - \partial_3 \alpha_2 \\ \partial_1 \alpha_1 - \partial_3 \alpha_3 \\ \partial_1 \alpha_2 + \partial_2 \alpha_3 \end{pmatrix}. \quad (4.5)$$

To the best of our knowledge, the operator $\llbracket \nabla \rrbracket_{\times_n}^T : C_c^\infty(\Omega, \mathbb{R}^{\frac{n(n-1)}{2}}) \rightarrow C_c^\infty(\Omega, \mathbb{R}^n)$ has not attracted attention in the literature so far, also not in coordinates. However, that differential operator plays the counterpart in the partial integration formula for the generalized curl_n , see (4.29a) below. The further differential operator appears here since the associated matrix to the generalized cross product has no symmetry.

Furthermore, it is the matrix representations of the cross product which allows us to introduce also a row-wise generalized matrix curl operator:

$$\operatorname{Curl}_n P := P \times_n (-\nabla) \stackrel{(3.23a)}{=} P \llbracket \nabla \rrbracket_{\times_n}^T \quad \text{for } P \in C_c^\infty(\Omega, \mathbb{R}^{m \times n}), \quad (4.6)$$

which is connected to the the column-wise differential operation:

$$\nabla \times_n B := \llbracket \nabla \rrbracket_{\times_n} B \stackrel{(3.23c)}{=} [\operatorname{Curl}_n B^T]^T \quad \text{for } B \in C_c^\infty(\Omega, \mathbb{R}^{n \times m}), \quad (4.7)$$

and like in the three dimensional setting can be referred as Curl_n^T .

Moreover, the matrix representation of the curl operation offers also a further differential operator $(\cdot) \llbracket \nabla \rrbracket_{\times_n}$ for $(m \times \frac{n(n-1)}{2})$ -matrix fields:

$$\mathfrak{P} \llbracket \nabla \rrbracket_{\times_n} \in C_c^\infty(\Omega, \mathbb{R}^{m \times n}) \quad \text{for } \mathfrak{P} \in C_c^\infty(\Omega, \mathbb{R}^{m \times \frac{n(n-1)}{2}}), \quad (4.8)$$

i.e., the row-wise differentiation from (4.4), and again related by transposition also $[[\nabla]]_{\times_n}^T(\cdot)$ for an $\left(\frac{n(n-1)}{2} \times m\right)$ -matrix fields.

Surely, it follows from (3.25b):

$$\operatorname{curl}_n(\nabla\alpha) \equiv 0 \quad \text{for } \alpha \in C_c^\infty(\Omega, \mathbb{R}), \quad (4.9a)$$

or even

$$\operatorname{Curl}_n(Da) \equiv 0 \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^m), \quad (4.9b)$$

and from (3.25c):

$$\begin{aligned} \operatorname{Curl}_n(Da)^T &= -2 \cdot \operatorname{Curl}_n(\operatorname{sym} Da) = 2 \cdot \operatorname{Curl}_n(\operatorname{skew} Da) \\ &= [D \operatorname{curl}_n a]^T = 2 \cdot [D \mathbf{a}_n(\operatorname{skew} Da)]^T \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^n). \end{aligned} \quad (4.10)$$

And as analogue to the usual $\operatorname{div} \circ \operatorname{curl} \equiv 0$ we have in n -dimensions:

$$\operatorname{div} [[\nabla]]_{\times_n}^T \mathbf{a} \stackrel{(3.15b)}{\equiv} 0 \quad \text{for } \mathbf{a} \in C_c^\infty(\Omega, \mathbb{R}^{\frac{n(n-1)}{2}}). \quad (4.11)$$

We recall the

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^n$ be open. A linear homogeneous differential operator with constant coefficients $\mathcal{A} : C_c^\infty(\Omega, \mathbb{R}^m) \rightarrow C_c^\infty(\Omega, \mathbb{R}^N)$ is said to be *elliptic* if its symbol $\mathbb{A}(b) \in \operatorname{Lin}(\mathbb{R}^m, \mathbb{R}^N)$ is injective for all $b \in \mathbb{R}^n \setminus \{0\}$.

It follows, from $b \times b = 0$ for $b \in \mathbb{R}^3$ that the usual curl operator is not elliptic. Similarly, also the generalized curl_n is not elliptic. Since the kernel of $\begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{R}^2$ consists only of 0 for all $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$ the operator $[[\nabla]]_{\times_2}^T$ is elliptic. To see that $[[\nabla]]_{\times_n}^T$ is not elliptic for all $n \geq 3$ we consider

$$\left[\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \right]_{\times_3}^T \begin{pmatrix} \beta_3 \\ -\beta_2 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -\beta_2 & -\beta_3 & 0 \\ \beta_1 & 0 & -\beta_3 \\ 0 & \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \beta_3 \\ -\beta_2 \\ \beta_1 \end{pmatrix} = 0 \quad (4.12)$$

which gives the non-ellipticity of $[[\nabla]]_{\times_3}^T$ and the higher dimensional cases follow from the inductive structure.

4.1 Nye's formulas

Denoting by Curl the matrix curl operator related to the usual curl for vector fields in \mathbb{R}^3 , *Room's identity* (3.37a) reads

$$\operatorname{Curl}(\operatorname{Anti}(a)) = L(Da) \quad \text{and} \quad Da = L(\operatorname{Curl} \operatorname{Anti}(a)) \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^3), \quad (4.13a)$$

where $\Omega(\text{open}) \subseteq \mathbb{R}^3$ for a moment. More precisely, they read

$$\operatorname{Curl}(\operatorname{Anti}(a)) = \operatorname{div} a \cdot I_3 - (Da)^T \quad (4.13b)$$

and

$$Da = \frac{\operatorname{tr}(\operatorname{Curl} \operatorname{Anti}(a))}{2} \cdot I_3 - (\operatorname{Curl} \operatorname{Anti}(a))^T \quad (4.13c)$$

and are better known as *Nye's formulas* [18, eq.(7)]. Surely, (4.13a)₁ is not surprising at all, but (4.13a)₂ implies that the entries of the derivative of a skew-symmetric matrix field are linear combinations of the entries of the matrix curl:

$$D A = L(\text{Curl } A) \quad \text{for } A \in C_c^\infty(\Omega, \mathfrak{so}(3)). \quad (4.13d)$$

Returning to the higher dimensional case we conclude from (3.35) or (3.37b)

$$D a = L(\llbracket a \rrbracket_{\times_n}^T \llbracket \nabla \rrbracket_{\times_n}) \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^n), \quad (4.14)$$

and from (3.37c)

$$D a = L(\text{Curl}_n \llbracket a \rrbracket_{\times_n}) \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^n), n \geq 3. \quad (4.15)$$

Note, however, that the latter expression is (in general) not related to $\text{curl}_n a$. Finally, from (3.37d) we deduce

$$D \mathfrak{a}_n(A) = L(\text{Curl}_n A) \quad \text{for } A \in C_c^\infty(\Omega, \mathfrak{so}(n)). \quad (4.16)$$

which implies (4.13d) in all dimensions $n \geq 2$:

$$D A = L(\text{Curl}_n A) \quad \text{for } A \in C_c^\infty(\Omega, \mathfrak{so}(n)), \quad (4.17)$$

a relation which is usually deduced using coordinates.

4.2 Incompatibility operator

Moreover, for $P \in C_c^\infty(\Omega, \mathbb{R}^{n \times n})$ we consider the generalized incompatibility operator given by:

$$\mathbf{inc}_n P := \nabla \times_n P \times_n \nabla = - \llbracket \nabla \rrbracket_{\times_n} P \llbracket \nabla \rrbracket_{\times_n}^T \quad (4.18)$$

$$\stackrel{(3.38)}{=} - \left[\text{Curl}_n \left((\text{Curl}_n P)^T \right) \right]^T \in C_c^\infty(\Omega, \mathbb{R}^{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}}). \quad (4.19)$$

It has the properties known from the usual incompatibility operator in three dimensions, it follows namely from (3.39c)

$$\text{sym } \mathbf{inc}_n P = \mathbf{inc}_n \text{sym } P \quad \text{and} \quad \text{skew } \mathbf{inc}_n P = \mathbf{inc}_n \text{skew } P \quad (4.20)$$

and from (3.41) for $a \in C_c^\infty(\Omega, \mathbb{R}^n)$:

$$\mathbf{inc}_n D a = \mathbf{inc}_n (D a)^T = \mathbf{inc}_n (\text{sym } D a) = \mathbf{inc}_n (\text{skew } D a) \equiv 0. \quad (4.21)$$

Furthermore, for matrix fields $\mathfrak{P} \in C_c^\infty(\Omega, \mathbb{R}^{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}})$ we consider the new differential operation

$$\llbracket \nabla \rrbracket_{\times_n}^T \mathfrak{P} \llbracket \nabla \rrbracket_{\times_n} \in C_c^\infty(\Omega, \mathbb{R}^{n \times n}) \quad (4.22)$$

with similar properties to the generalized incompatibility operator, see section 3.11. Especially for $\zeta \in C_c^\infty(\Omega, \mathbb{R})$ we obtain:

$$\llbracket \nabla \rrbracket_{\times_n}^T \zeta \cdot I_{\frac{n(n-1)}{2}} \llbracket \nabla \rrbracket_{\times_n} \stackrel{(3.44)}{=} \Delta \zeta \cdot I_n - D \nabla \zeta, \quad (4.23)$$

where we have used that from an algebraic point of view $\Delta = \|\nabla\|^2$ behaves like a scalar and where $D \nabla \zeta$ is the Hessian matrix of ζ . The latter expression reminds of the known identity in $n = 3$ dimensions for the usual incompatibility operator:

$$\mathbf{inc}(\zeta \cdot I_3) = \Delta \zeta \cdot I_3 - D \nabla \zeta. \quad (4.24)$$

It is clear from the integration by parts formula for the generalized curl (4.29b), that and how the operator $\llbracket \nabla \rrbracket_{\times_n}^T (\cdot) \llbracket \nabla \rrbracket_{\times_n}$ will play the counterpart in the corresponding integration by parts formula for the generalized incompatibility operator. For the corresponding formula in the usual case we refer the reader to [1].

Remark 4.2. The usual incompatibility operator \mathbf{inc} occurs, e.g., in the modeling of dislocated crystals or in the modeling of elastic materials with dislocations, where the notion of incompatibility is at the basis of a new paradigm to describe the inelastic effects, see e.g. [1, 10, 4, 16]. The coordinate-free view presented above should provide a better understanding of such phenomena also in higher dimensions.

4.3 Vector Laplacian

Recalling (3.33) we have for all $a, b \in \mathbb{R}^n$:

$$\llbracket b \rrbracket_{\times_n}^T b \times_n a = \llbracket b \rrbracket_{\times_n}^T \llbracket b \rrbracket_{\times_n} a \stackrel{(3.33)}{=} \|b\|^2 \cdot a - b \cdot \langle b, a \rangle. \quad (4.25)$$

Thus, interchanging b by ∇ we deduce

$$\Delta a = \nabla \operatorname{div} a + \llbracket \nabla \rrbracket_{\times_n}^T \operatorname{curl}_n a \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^n), n \geq 2, \quad (4.26)$$

which is the generalization of the known expression for the vector Laplacian in $n = 3$ dimensions:

$$\Delta a = \nabla \operatorname{div} a - \operatorname{curl} \operatorname{curl} a \quad \text{for } a \in C_c^\infty(\Omega, \mathbb{R}^3), \quad (4.27)$$

and the appearance of the minus sign comes from the fact, that the associated matrix with the usual cross product is a skew-symmetric matrix.

Since the matrix divergence and matrix curl act row-wise, we obtain

$$\Delta P = \operatorname{D} \operatorname{Div} P + (\operatorname{Curl}_n P) \llbracket \nabla \rrbracket_{\times_n} \quad \text{for } P \in C_c^\infty(\Omega, \mathbb{R}^{m \times n}), \quad (4.28)$$

for $m, n \in \mathbb{N}$, $n \geq 2$, meaning that the entries of the Laplacian of a matrix field P are linear combinations of the entries of the derivative of the matrix curl and of the entries of the derivative of the matrix divergence.

4.4 Integration by parts

For the sake of completeness we include the integration by parts formula for the generalized matrix curl: Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary $\partial\Omega$ and outward unit normal ν . For all $a \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and all $\mathbf{a} \in C^1(\overline{\Omega}, \mathbb{R}^{\frac{n(n-1)}{2}})$ we have

$$\int_{\Omega} \langle \operatorname{curl}_n a, \mathbf{a} \rangle + \langle a, \llbracket \nabla \rrbracket_{\times_n}^T \mathbf{a} \rangle dx = \int_{\partial\Omega} \langle a \times_n (-\nu), \mathbf{a} \rangle dS, \quad (4.29a)$$

so that for matrix fields $P \in C^1(\overline{\Omega}, \mathbb{R}^{m \times n})$ and $\mathfrak{P} \in C^1(\overline{\Omega}, \mathbb{R}^{m \times \frac{n(n-1)}{2}})$ it follows

$$\int_{\Omega} \langle \operatorname{Curl}_n P, \mathfrak{P} \rangle + \langle P, \mathfrak{P} \llbracket \nabla \rrbracket_{\times_n} \rangle dx = \int_{\partial\Omega} \langle P \times_n (-\nu), \mathfrak{P} \rangle dS, \quad (4.29b)$$

and we refer the reader to [15] for a coordinate-free proof for square matrix fields P .

4.5 Helmholtz decomposition

It is well known, that any vector field $a \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ admits a decomposition into a divergence-free vector field and a gradient field, i.e., a curl_n -free part, see e.g. [5] and for a deviation from the Hodge decomposition see [9]. Let us denote the divergence-free part by a_{div} and the curl_n -free by $a_{\operatorname{curl}_n}$, so having $a = a_{\operatorname{div}} + a_{\operatorname{curl}_n}$. At the end of our vector calculus we will provide the reader with the explicit coordinate-free expressions of those parts, thus, providing the Helmholtz decomposition explicitly in all dimensions $n \geq 2$. More precisely, we show that

$$a_{\operatorname{curl}_n}(x) = \nabla_x \int_{\mathbb{R}^n} G^{(n)}(x, y) \cdot \operatorname{div} a(y) dy \quad (4.30a)$$

$$= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} (x - y) \cdot (\|x - y\|^{-n} \operatorname{div} a(y)) dy, \quad (4.30b)$$

and

$$a_{\text{div}}(x) = \llbracket \nabla_x \rrbracket_{\times_n}^T \int_{\mathbb{R}^n} G^{(n)}(x, y) \cdot \text{curl}_n a(y) \, dy \quad (4.30c)$$

$$= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \llbracket x - y \rrbracket_{\times_n}^T (\|x - y\|^{-n} \cdot \text{curl}_n a(y)) \, dy, \quad (4.30d)$$

where $G^{(n)}(x, y)$ denotes the normalized *fundamental Green's function for the Laplacian* for the entire space \mathbb{R}^n and is given by

$$G^{(n)}(x, y) = \begin{cases} \frac{1}{2\pi} \ln \|x - y\|, & \text{for } n = 2, \\ \frac{1}{n(2-n)\omega_n} \|x - y\|^{2-n}, & \text{for } n \geq 3, \end{cases} \quad (4.31)$$

denoting by ω_n the volume of the unit ball in \mathbb{R}^n , see [7, Section 2.4]. Indeed, the first expressions in (4.30) follows from the decomposition (4.26) since for $a \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ we have

$$\begin{aligned} a(x) &= \int_{\mathbb{R}^n} a(y) \cdot \Delta_x G^{(n)}(x, y) \, dy = \Delta_x \int_{\mathbb{R}^n} a(y) \cdot G^{(n)}(x, y) \, dy \\ &\stackrel{(4.26)}{=} \nabla_x \text{div}_x \int_{\mathbb{R}^n} a(y) \cdot G^{(n)}(x, y) \, dy + \llbracket \nabla_x \rrbracket_{\times_n}^T \text{curl}_{n,x} \int_{\mathbb{R}^n} a(y) \cdot G^{(n)}(x, y) \, dy \\ &= \nabla_x \int_{\mathbb{R}^n} \langle a(y), \nabla_x G^{(n)}(x, y) \rangle \, dy + \llbracket \nabla_x \rrbracket_{\times_n}^T \int_{\mathbb{R}^n} \nabla_x G^{(n)}(x, y) \times_n a(y) \, dy \\ &\stackrel{(*)}{=} \nabla_x \int_{\mathbb{R}^n} \langle a(y), -\nabla_y G^{(n)}(x, y) \rangle \, dy + \llbracket \nabla_x \rrbracket_{\times_n}^T \int_{\mathbb{R}^n} a(y) \times_n \nabla_y G^{(n)}(x, y) \, dy \\ &\stackrel{(**)}{=} \nabla_x \int_{\mathbb{R}^n} G^{(n)}(x, y) \cdot \text{div} a(y) \, dy + \llbracket \nabla_x \rrbracket_{\times_n}^T \int_{\mathbb{R}^n} G^{(n)}(x, y) \cdot \text{curl}_n a(y) \, dy, \end{aligned}$$

where in (*) we used that $\nabla_x G^{(n)}(x, y) = -\nabla_y G^{(n)}(x, y)$ and in (**) the relations

$$\text{div}(\alpha \cdot a) = \langle \nabla \alpha, a \rangle + \alpha \text{div} a \quad \text{and} \quad \text{curl}_n(\alpha \cdot a) = \nabla \alpha \times_n a + \alpha \cdot \text{curl}_n a, \quad (4.32)$$

for $\alpha \in C_c^\infty(\Omega)$ and $a \in C_c^\infty(\Omega, \mathbb{R}^n)$. Since we have

$$\nabla_x G^{(n)}(x, y) = \frac{1}{n\omega_n} \|x - y\|^{-n} \cdot (x - y) \quad \text{for } n \geq 2, \quad (4.33)$$

we obtain

$$a_{\text{curl}_n}(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} (x - y) \cdot (\|x - y\|^{-n} \text{div} a(y)) \, dy, \quad (4.34a)$$

$$a_{\text{div}}(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \llbracket x - y \rrbracket_{\times_n}^T (\|x - y\|^{-n} \cdot \text{curl}_n a(y)) \, dy, \quad (4.34b)$$

and end up with *Riesz potentials* of order 1, see [20, Section V.1].

5 Conclusion

In the present paper we investigated in the algebraic structures underlying the generalized cross product by associating it with an adequate matrix multiplication. The situation is different from the case of the usual cross product where a matrix representation yields a skew-symmetric matrix. The absence of symmetry in the general case causes to adjust the known algebraic identities in an adequate manner and include also other combinations. In the vector calculus this returned not only the generalized curl_n but also a new operator $\llbracket \nabla \rrbracket_{\times_n}^T$. The importance of the latter has been emphasized in the previous section and,

especially, by the fact that the image of the $[\nabla]_{\times^n}^T$ operator lies in the kernel of the divergence operator, see (4.11). Here, we have carefully investigated in the matrix analysis behind such operations. Such a view already turned out to be very fruitful in extending Korn inequalities for incompatible tensor fields to higher dimensions, cf. [15] where first thoughts in that matrix representations have been investigated. With the better understanding presented here we are now in a position to further extend Korn-Maxwell-Sobolev type inequalities which will be the subject of a forthcoming paper.

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